

**UNCLASSIFIED**

**AD 410256**

**DEFENSE DOCUMENTATION CENTER**

**FOR**

**SCIENTIFIC AND TECHNICAL INFORMATION**

**CAMERON STATION, ALEXANDRIA, VIRGINIA**



**UNCLASSIFIED**

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

⑤ 17503

①

See 1

AD N410256  
DDC FILE COPY  
410256  
④ 1.60

# A Preliminary Study of the Fields in a Rectangular Waveguide with an Oblique Shorted End

14 JUNE 1963

*Prepared by* HERMAN A. LANG  
*Plasma Research Laboratory*

*Prepared for* COMMANDER SPACE SYSTEMS DIVISION  
UNITED STATES AIR FORCE  
*Inglewood, California*

RECEIVED  
JUL 17 1963  
TISA A



LABORATORIES DIVISION • AEROSPACE CORPORATION  
CONTRACT NO. AF 04(695)-169

SSD-TDR 63-131

Report No.

TDR/169(3230-22)TN/3

⑦ DNA  
E. coli  
⑧ DNA  
HSP  
⑨ DNA  
L. casei  
⑩ DNA  
H. u.  
⑪ DNA

**AEROSPACE CORPORATION**  
El Segundo, California

Contract No. AF 04(695)-169

14 June 1963 ,

Prepared for  
COMMANDER SPACE SYSTEMS DIVISION  
UNITED STATES AIR FORCE  
Inglewood, California

# ABSTRACT

*It is shown*  
~~This study shows~~ that if the fundamental mode in a rectangular wave guide undergoes a phase shift  $2\pi$  <sup>*lambda*</sup> when it encounters a sloping shorted end, each evanescent mode of the near field at the shorted end possesses a common phase angle  $(3\pi/2) - \pi$  <sup>*lambda*</sup>.  
The equations which determine the relations between the phase <sup>*sub n*</sup> and the amplitudes <sup>*lambda*</sup> of the evanescent modes are established. The results suggested that <sup>*lambda*</sup> and <sup>*sub n*</sup> are complicated functions of the angle of slope. Some experimental work is recommended if further progress is to be made.

## CONTENTS

I.	INTRODUCTION . . . . .	1
II.	BASIC EQUATIONS . . . . .	3
III.	ANALYSIS OF THE PHASE . . . . .	9
IV.	CONCLUDING COMMENTS . . . . .	15

## FIGURES

1	Oblique Interfaces in Rectangular Waveguides . . .	2
2	Phase Angles on the Unit Circle . . . . .	12

## I. INTRODUCTION

This preliminary study considers the reflection of an incident wave from the oblique shorted end of a rectangular waveguide (Fig. 1a). The motivation for the investigation was prompted by the desire to obtain reflection and transmission coefficients for an oblique slab (Fig. 1c) having a negative dielectric constant  $\epsilon_1$ . A complex dielectric constant,  $\epsilon_1$ , is also of interest since the slab would macroscopically represent a plasma. A knowledge of reflection and transmission characteristics would be useful in attempting to use microwaves for plasma diagnostics. A contained plasma differs markedly from a uniform slab but the effects of obliquity are still of interest.

It was surprising to find that, despite the wealth of papers on microwaveguide propagation, the discussion of oblique incidence appears to be limited to plane EM waves. Indeed, an extensive library search failed to disclose any reference to oblique reflections in waveguides.

Since it can be demonstrated that no combination of plane EM waves obliquely incident on a slab or interface can be combined in such a way that the boundary conditions at the walls of the waveguide are satisfied, it appears that the effect of obliquity is to produce complicated mode conversion. To examine this in more detail, we consider the problem shown in Fig. 1a rather than that shown in Fig. 1b.

This eliminates any transmitted waves. In fact, the principle conclusion of this note is that all evanescent modes possess a common phase which, in turn, determines a phase shift of the single allowed propagating mode. This conclusion depends directly on the fact that energy is conserved in the lossless guide and would not be correct when transmitted waves are present.

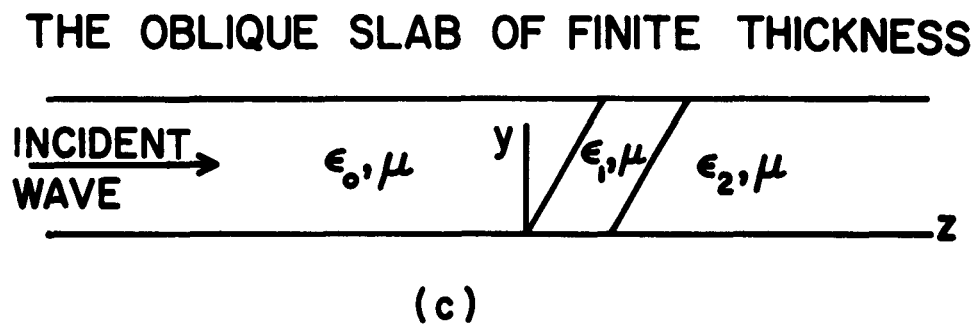
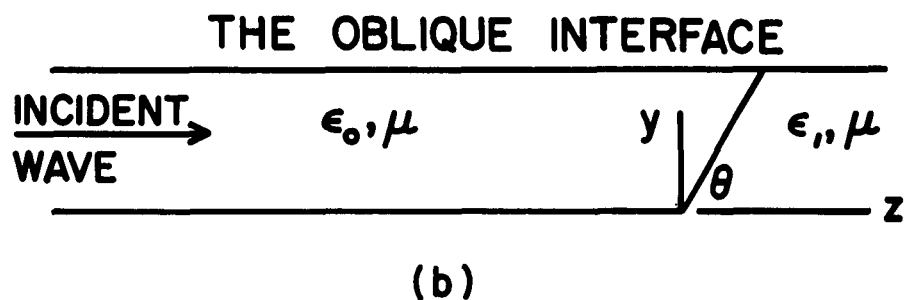
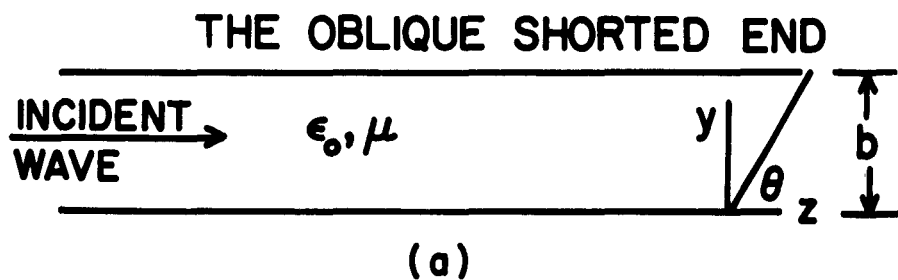


Fig. 1. Oblique Interfaces in Rectangular Waveguides



## II. BASIC EQUATIONS

For the right-hand set of axes  $xyz$ , the origin of coordinates is such that  $z = 0$  locates the lower edge of the slab. The wave is incident from the left and the waveguide dimensions  $a, b$  are chosen so that only the lowest mode propagates. For the  $H_0$  or  $TE_0$  mode and a harmonic time factor  $e^{-i\omega t}$ , the wave incident in free space is represented by the electric and magnetic field components

$$E_x = A \sin qy e^{i\Gamma z}$$

$$H_y = -\frac{1}{i\omega\mu} \frac{\partial E_x}{\partial z} = \frac{-\Gamma}{\omega\mu} (A \sin qy e^{i\Gamma z})$$

$$H_z = \frac{1}{i\omega\mu} \frac{\partial E_x}{\partial y} = \frac{q}{i\omega\mu} (A \cos qy e^{i\Gamma z})$$

where Maxwell's equations for free space reduce to  $\nabla \cdot \vec{H} = \nabla \cdot \vec{E} = 0$  and  $\nabla \times \vec{E} = i\omega\mu \vec{H}$ ;  $\nabla \times \vec{H} = -i\omega\epsilon \vec{E}$ . The first two of these are identically satisfied. The third equation was used to derive  $H_y, H_z$  from  $E_x$ . The last equation reduces to the wave equation

$$\left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) E_x = 0$$

where  $k^2 = \omega^2 \epsilon_0 \mu$ .

From the wave equation,  $q^2 + \Gamma^2 = k^2$ . For homogeneous isotropic media, the field quantities  $\vec{B} = \mu \vec{H}$  and  $\vec{D} = \epsilon \vec{E}$  are not required. The requirement for a wave propagating to the right is  $\Gamma$  real and  $> 0$ . The reflected wave field corresponds to  $\Gamma$  real and  $< 0$ .

The boundary conditions on the walls of the guide reduce to  $E_x = 0$ , for  $y = 0, b$ , which can be satisfied by choosing  $qb = n\pi$  where  $n = 1.0$  for the lowest ( $H_0$  or  $TE_0$ ) mode

For any mode,  $q_n = n\pi/b$  and the corresponding wave number in the  $z$  direction is:

$$\Gamma = \sqrt{k^2 - \frac{n^2 \pi^2}{b^2}} = \frac{\pi}{b} \sqrt{\left(\frac{bk}{\pi}\right)^2 - n^2}$$

In the  $x$  direction, the guide dimension  $a$  is selected to allow only this mode of operation. To maintain this mode requires that

$$1 < a = \left(\frac{bk}{\pi}\right)^2 < 4$$

in order that  $\Gamma$  is real. For  $n \geq 2$ ,  $\Gamma$  becomes imaginary corresponding to evanescent modes.

It is convenient to introduce the dimensionless length  $s = \pi y/b = qy$  so the incident wave field of unit amplitude is:

$$E_x = \sin se^{iz \frac{\pi}{b} \sqrt{a-1}}$$

$$H_y = \frac{-\pi}{\omega \mu b} \sqrt{a-1} \sin se^{iz \frac{\pi}{b} \sqrt{a-1}}$$

$$H_z = \frac{-i\pi}{b\omega \mu} \cos se^{iz \frac{\pi}{b} \sqrt{a-1}}$$

The reflected wave field is:

$$E_x = A_1 \sin se^{-iz \frac{\pi}{b} \sqrt{a-1}}$$

$$H_y = \frac{\pi}{\omega \mu b} A_1 \sqrt{a-1} \sin se^{-iz \frac{\pi}{b} \sqrt{a-1}}$$

$$H_z = \frac{-i\pi}{b\omega\mu} A_1 \cos se^{-iz \frac{\pi}{b} \sqrt{a-1}}$$

For higher evanescent fields, we have

$$E_x^n = A_n \sin nse^{z \frac{\pi}{b} \sqrt{n^2-a}}$$

$$H_y^n = \frac{in\pi \sqrt{n^2-a}}{\omega \mu b} A_n \sin nse^{z \frac{\pi}{b} \sqrt{n^2-a}}$$

$$H_z^n = \frac{-in\pi}{b\omega\mu} A_n \cos nse^{z \frac{\pi}{b} \sqrt{n^2-a}}$$

The positive square root is to be used throughout and the sign of the  $z$  term of the evanescent fields is determined by the requirement that the fields decay to the left.

It remains to satisfy the boundary conditions on the shorted end which reduce to

$$\left\{ \begin{array}{l} E_x = 0 \\ H_z \sin \theta - H_y \cos \theta = 0 \end{array} \right\} \quad \text{for } z = Ky, \quad 0 < y < b$$

where  $K = \cos \theta$ . At normal incidence,  $K = 0$ . The two boundary equations become:

$$\begin{aligned}
0 &= \sin s \left( e^{iKs\sqrt{a-1}} + A_1 e^{-iKs\sqrt{a-1}} \right) + \sum_2^{\infty} A_n \sin nse^{Ks\sqrt{n^2-a}} \\
0 &= iK \sin s\sqrt{a-1} \left( e^{iKs\sqrt{a-1}} - A_1 e^{-iKs\sqrt{a-1}} \right) \\
&+ K \sum_2^{\infty} A_n \sin ns\sqrt{n^2-a} e^{Ks\sqrt{n^2-a}} \\
&+ \left[ \cos s \left( e^{iKs\sqrt{a-1}} + A_1 e^{-iKs\sqrt{a-1}} \right) \right. \\
&\quad \left. + \sum_2^{\infty} nA_n \cos nse^{Ks\sqrt{n^2-a}} \right]
\end{aligned}$$

The second boundary equation is obtained by discarding the constant factor  $-i\pi/b\omega\mu \sin \theta$  and forming  $(H_z - KH_y) = 0$ . Differentiation of the first boundary condition leads directly to the second boundary equation. For normal incidence,  $K = 0$  and all evanescent modes vanish. Both equations reduce to  $1 + A_1 = 0$  so the reflection coefficient is simply  $A_1 = -1.0 = e^{i\pi}$  and the phase of the reflected wave is  $180^\circ$ .

When there is oblique incidence, the absence of transmission in the problem means that energy is conserved, and, consequently, far to the left, the reflected amplitude continues to be 1.0. The effect of the shorted end

produces only a phase shift of the lowest reflected mode. It is convenient to write this in the form

$$A_1 = e^{i(\pi-2\lambda)} = -e^{-2i\lambda}$$

### III. ANALYSIS OF THE PHASE

Rewriting the boundary equations with  $A_1 = -e^{-2i\lambda}$  in the form

$$\begin{aligned}
 -\sin s \left[ e^{-iKs\sqrt{a-1}} - e^{-i(Ks\sqrt{a-1}+2\lambda)} \right] &= \sum_2^{\infty} A_n \sin nse^{Ks\sqrt{n^2-a}} \\
 -iK \sin s\sqrt{a-1} \left[ e^{iKs\sqrt{a-1}} + e^{-i(Ks\sqrt{a-1}+2\lambda)} \right] - \cos s \left[ e^{iKs\sqrt{a-1}} - e^{-i(Ks\sqrt{a-1}+2\lambda)} \right] \\
 &= \sum_2^{\infty} A_n \sin nsK\sqrt{n^2-a} e^{Ks\sqrt{n^2-a}} + \sum_2^{\infty} nA_n \cos nse^{Ks\sqrt{n^2-a}}
 \end{aligned}$$

suggests the further reduction

$$\begin{aligned}
 -\sin s(2ie^{-i\lambda})[\sin(Ks\sqrt{a-1} + \lambda)] &= \sum_2^{\infty} A_n \sin nse^{Ks\sqrt{n^2-a}} \\
 -iK \sin s(2e^{-i\lambda})\sqrt{a-1} [\cos(Ks\sqrt{a-1} + \lambda)] - \cos s(2ie^{-i\lambda})[\sin(Ks\sqrt{a-1} + \lambda)] \\
 &= \sum_2^{\infty} A_n n \cos nse^{Ks\sqrt{n^2-a}} + \sum_2^{\infty} A_n \sin nsK\sqrt{n^2-a} e^{Ks\sqrt{n^2-a}}
 \end{aligned}$$

The terms on the left of both equations represent the propagating modes and the ratio of real to imaginary terms is  $\tan \lambda$ , independent of  $s$ . The complex amplitudes of the evanescent modes  $A_n$  on the right must have the same structure if the sum on the right is to equal the terms on the left of both equations.

It follows that every evanescent mode possesses the same direction.  
Writing

$$A_n = \rho_n e^{i\pi + i\frac{\pi}{2} - i\lambda} = -\rho_n i e^{-i\lambda}$$

the boundary conditions reduce to

$$\begin{aligned} 2 \sin s \sin (Ks\sqrt{a-1} + \lambda) &= \sum_2^{\infty} \rho_n \sin nse^{Ks\sqrt{n^2-a}} \\ 2 \sin s(K\sqrt{a-1}) \cos (Ks\sqrt{a-1} + \lambda) &+ 2 \cos s \sin (Ks\sqrt{a-1} + \lambda) \\ &= \sum_2^{\infty} \rho_n K\sqrt{n^2-a} \sin nse^{Ks\sqrt{n^2-a}} + \sum_2^{\infty} \rho_n n \cos nse^{Ks\sqrt{n^2-a}} \end{aligned}$$

The interval  $0 < s < \pi$  may be divided into  $n$  intervals and the above sums truncated to construct  $n$  simultaneous equations for determining the amplitudes  $\rho_n$  and the phase angle  $\lambda$ . As  $n \rightarrow \infty$ , we approach a generally unsolvable set of equations such that each  $\rho_n$  is the ratio of two infinite determinants. The structure of these determinants can be examined to lead to the same conclusion derived above--that all evanescent modes have the same phase angle but for a factor  $e^{i\pi}$ . The result depends only on the fact that the elements of the determinant are real for the evanescent modes while the elements of the propagating modes are complex. To clarify this statement, the structure of these equations can be used to show that

$$A_2 = \frac{-iC}{A + iB}$$

$$A_3 = \frac{-iD}{A + iB}$$

etc.

where  $\tan \lambda = B/A$ . Here we know only that A, B, C, D are real but this is sufficient to determine that the phase of  $A_2, A_3, \dots, A_n$  is  $\phi_n$  where  $\tan \phi_n = A/B = \cot \lambda$ .

The simplest way to represent this result is in terms of the Argand-type diagram shown in Fig. 2. Note that since some evanescent amplitudes may be negative the phase is either  $(\pi/2)-\lambda$  or  $(3\pi/2)-\lambda$ . The diagram also generates the equation

$$\sum_2^{\infty} \rho_n = 2 \sin \lambda$$

It is not possible to determine by a physical argument which evanescent mode has the phase  $(\pi/2)-\lambda$  or  $(3\pi/2)-\lambda$ , but preliminary calculations strongly suggest that the phase angle is, in fact,  $(3\pi/2)-\lambda$ . In Fig. 2, this means all vectors  $\vec{A}_n$  are directed from A to B and is equivalent to the assumption that the amplitudes  $\rho_n$  are all positive. However, this assumption is not utilized in the subsequent steps.

The removal of the phase angles leads to the two equations

$$2 \sin s \sin (Ks\sqrt{a-1} + \lambda) = \sum_2^{\infty} \rho_n \sin nse^{Ks\sqrt{n^2-a}}$$

$$2[K\sqrt{a-1} \sin s \cos (Ks\sqrt{a-1} + \lambda) + \cos s \sin (Ks\sqrt{a-1} + \lambda)]$$

$$= \sum_2^{\infty} \rho_n \left[ \sin ns \left( K\sqrt{n^2-a} \right) + n \cos ns \right] e^{Ks\sqrt{n^2-a}}$$





where all quantities are real. It is evident that  $\rho_n$ ,  $\lambda$  are functions only of  $K$ ,  $a$  and we are left with the vastly more difficult problem of determining these quantities. To remove the  $y$  dependence, we integrate the equations with respect to  $s$  from 0 to  $\pi$ .

The integrated equations are:

$$\frac{2[\sin \lambda + \sin (K\pi\sqrt{a-1} + \lambda)]}{1 - K^2(a-1)} = \sum_2^{\infty} \frac{\rho_n n}{n^2 + K^2(n^2 - a)} \left[ 1 - (-1)^n e^{K\pi\sqrt{n^2 - a}} \right]$$

and

$$0 = \sum_2^{\infty} \frac{\rho_n (n - K\sqrt{n^2 - a})}{n^2 + K^2(n^2 - a)} \left[ 1 - (-1)^n e^{K\pi\sqrt{n^2 - a}} \right]$$

to these we may add the relation

$$2 \sin \lambda = \sum_2^{\infty} \rho_n$$

The left side of the section equation vanishes because  $E_x$  vanishes at the wall.

#### IV. CONCLUDING COMMENTS

The three equations obtained represent the conclusions of this phase of the study. As we wish to find  $\lambda(K, \alpha)$  and  $\rho_n(K, \alpha)$ , it appears desirable to conduct experiments with a number of waveguides of different degrees of obliquity. This would determine directly the phase shift  $\lambda$  as a function of angle of obliquity  $\theta = \cot^{-1} K$  for a given medium ( $\alpha$  a fixed constant).

We may in this way obtain the clues needed to construct integral representations that are capable of representing both members of the three equations; that is, a knowledge of  $\lambda(K)$  might indicate how to establish a contour integral possessing residue terms that generate the sums and uniquely determine all  $\rho_n(K)$ . It is difficult to see how to proceed without such information because of the branch points  $n = \pm \sqrt{\alpha}$  that exist. It is probable that each  $\rho_n$  is an extremely complicated function of the parameters  $n, K, \alpha$ . The choice of particular numerical values for  $K$  or  $\alpha$  does not materially simplify the equations.

The results of the attempts made to date to solve the three equations for  $\lambda, \rho_n$  as functions of  $K, \alpha$  are not promising, but do suggest some helpful hints. It appears desirable to defer further results or discussion of approximate procedures to a subsequent report.

This phase of the study may be summarized as follows: The effect of obliquity at the shorted end of a waveguide produces a phase shift in the single propagating E mode which is allowed by the guide dimensions. The magnitude of the phase shift  $2\lambda$  is determined by the amplitudes of the evanescent modes of the field near the sloping end. Both these amplitudes and the phase shift are complicated functions of the angle of slope. All evanescent modes possess the common phase angle,  $(3\pi/2) - \lambda$ .

UNCLASSIFIED	<p>Aerospace Corporation, El Segundo, California. A PRELIMINARY STUDY OF THE FIELDS IN A RECTANGULAR WAVEGUIDE WITH AN OBLIQUE SHORTED END, prepared by Herman A. Lang. 14 June 1963. [22]p. incl. illus. (Report TDR-169(3230-22)TN-3;SSD-TDR-63-131) (Contract AF 04(695)-169) Unclassified report</p> <p>This study shows that if the fundamental mode in a rectangular wave guide undergoes a phase shift <math>2\lambda</math> when it encounters a sloping shorted end, each evanescent mode of the near field at the shorted end possesses a common phase angle <math>(3\pi/2)-\lambda</math>. The equations which determine the relations between the phase <math>\lambda</math> and the amplitudes <math>p_n</math> of the evanescent modes are established. The results suggested that <math>\lambda</math> and <math>p_n</math> are complicated functions of the angle of slope. Some experimental work is recommended if further progress is to be made.</p>
--------------	--

UNCLASSIFIED	<p>Aerospace Corporation, El Segundo, California. A PRELIMINARY STUDY OF THE FIELDS IN A RECTANGULAR WAVEGUIDE WITH AN OBLIQUE SHORTED END, prepared by Herman A. Lang. 14 June 1963. [22]p. incl. illus. (Report TDR-169(3230-22)TN-3;SSD-TDR-63-131) (Contract AF 04(695)-169) Unclassified report</p> <p>This study shows that if the fundamental mode in a rectangular wave guide undergoes a phase shift <math>2\lambda</math> when it encounters a sloping shorted end, each evanescent mode of the near field at the shorted end possesses a common phase angle <math>(3\pi/2)-\lambda</math>. The equations which determine the relations between the phase <math>\lambda</math> and the amplitudes <math>p_n</math> of the evanescent modes are established. The results suggested that <math>\lambda</math> and <math>p_n</math> are complicated functions of the angle of slope. Some experimental work is recommended if further progress is to be made.</p>
--------------	--

UNCLASSIFIED	<p>Aerospace Corporation, El Segundo, California. A PRELIMINARY STUDY OF THE FIELDS IN A RECTANGULAR WAVEGUIDE WITH AN OBLIQUE SHORTED END, prepared by Herman A. Lang. 14 June 1963. [22]p. incl. illus. (Report TDR-169(3230-22)TN-3;SSD-TDR-63-131) (Contract AF 04(695)-169) Unclassified report</p> <p>This study shows that if the fundamental mode in a rectangular wave guide undergoes a phase shift <math>2\lambda</math> when it encounters a sloping shorted end, each evanescent mode of the near field at the shorted end possesses a common phase angle <math>(3\pi/2)-\lambda</math>. The equations which determine the relations between the phase <math>\lambda</math> and the amplitudes <math>p_n</math> of the evanescent modes are established. The results suggested that <math>\lambda</math> and <math>p_n</math> are complicated functions of the angle of slope. Some experimental work is recommended if further progress is to be made.</p>
--------------	--

UNCLASSIFIED	<p>Aerospace Corporation, El Segundo, California. A PRELIMINARY STUDY OF THE FIELDS IN A RECTANGULAR WAVEGUIDE WITH AN OBLIQUE SHORTED END, prepared by Herman A. Lang. 14 June 1963. [22]p. incl. illus. (Report TDR-169(3230-22)TN-3;SSD-TDR-63-131) (Contract AF 04(695)-169) Unclassified report</p> <p>This study shows that if the fundamental mode in a rectangular wave guide undergoes a phase shift <math>2\lambda</math> when it encounters a sloping shorted end, each evanescent mode of the near field at the shorted end possesses a common phase angle <math>(3\pi/2)-\lambda</math>. The equations which determine the relations between the phase <math>\lambda</math> and the amplitudes <math>p_n</math> of the evanescent modes are established. The results suggested that <math>\lambda</math> and <math>p_n</math> are complicated functions of the angle of slope. Some experimental work is recommended if further progress is to be made.</p>
--------------	--